M13/5/FURMA/SP2/ENG/TZ0/XX/M

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MARKSCHEME

May 2013

FURTHER MATHEMATICS

Standard Level

Paper 2

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2013**". It is **essential** that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)****.*

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT****.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) (i) the mode is 1 ***A1***

(ii) attempt to solve $\frac{1-p}{p^2} = \frac{28}{9}$ ***M1***

obtain $p = \frac{3}{7}$ ***A1***

Note: $p = 0.429$ is awarded ***M1A0***.

[3 marks]

(b) (i) require least n such that

$0.55^n < 0.01$ ***(M1)***

EITHER

listing values: 0.55, 0.3025, 0.166, 0.091, 0.050, 0.028, 0.015, 0.0084 ***(M1)***

obtain $n = 8$ ***A1***

OR

$n > \frac{\ln 0.01}{\ln 0.55} = 7.70\ldots$ ***(M1)***

obtain $n = 8$ ***A1***

(ii) recognition of negative binomial ***(M1)***

$X \sim NB(3,\ 0.45)$

$\mathrm{P}(X = 8) = \binom{7}{2} \times 0.45^3 \times 0.55^5$ ***(A1)***

$= 0.0963$ ***A1***

Note: If 0.45 and 0.55 are mixed up in (b), count it as a misread – probability in that case is 0.0645.

[6 marks]

Total [9 marks]

2. (a) (i) $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$ ***(M1)A1***

$\displaystyle\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{N-1} - \frac{1}{N} + \frac{1}{N} - \frac{1}{N+1}$ ***A1***

$1 - \dfrac{1}{N+1}$ ***M1***

the series tends to a finite limit (1) as $N \to \infty$ ***R1***

hence the series converges ***AG***

Note	The second ***M1*** is for evidence of the telescoping of the series, even if summed to infinity.

(ii) consider the ratio of n^{th} terms ***(M1)***

$\dfrac{\frac{e^{-n}}{n^2}}{\frac{1}{n(n+1)}} = \dfrac{(n+1)e^{-n}}{n}$ ***A1***

$\to 1 \times 0 = 0$ as $n \to \infty$ ***A1***

so the series converges ***R1***

[9 marks]

(b) (i) consider $\displaystyle\int_0^R \frac{1}{x^2+1}\mathrm{d}x$ ***M1***

$= \left[\arctan(x)\right]_0^R = \arctan(R)$ ***A1***

$\displaystyle\lim_{R\to\infty} \arctan(R) = \frac{\pi}{2}$ (a finite number) ***R1***

hence the improper integral is convergent ***AG***

(ii) the terms of the series are positive ***A1***
the terms are decreasing ***A1***
the terms tend to zero ***A1***
by the integral test, the series converges ***AG***

[6 marks]

continued …

Question 2 continued

(c) (i) the absolute values of the terms are monotonically decreasing ***A1***
to zero ***A1***
the series converges by the alternating series test ***R1AG***

Note: Accept absolute convergence, with reference to part (b)(ii) $\Rightarrow$ convergence.

(ii) statement that successive partial sums bound the total sum ***R1***

$S > \frac{1}{1} - \frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{3}{5}$ ***A1***

$S < \frac{1}{1} - \frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \frac{1}{17} = 0.6588$ ***A1***

$S < 0.6588 < \frac{2}{3}$ ***AG***

[6 marks]

(d) (i) consider $\left| \frac{\frac{x^{n+1}}{(n+1)^2+1}}{\frac{x^n}{n^2+1}} \right|$ ***M1***

$= \left| \frac{x(n^2+1)}{(n+1)^2+1} \right|$ ***A1***

$\to |x| \text{ as } n \to \infty$ ***A1***

therefore radius of convergence $= 1$ ***A1***

(ii) interval of convergence $= [-1, 1]$ ***A1A1***

Note: ***A1*** for $[-1$, and ***A1*** for $1]$.

[6 marks]

Total [27 marks]

3. (a) (i) $\mathrm{P}(X > a) = \int_a^{\infty} \lambda e^{-\lambda x}\,\mathrm{d}x$ ***M1***

$[-e^{-\lambda x}]_a^{\infty}$ ***A1***

$= e^{-\lambda a}$ ***A1***

(ii) $\mathrm{P}(X > 10) = e^{-0.3}\ (= 0.74\ldots)$ ***(M1)A1***

(iii) probability of a safe double crossing $= e^{-0.6}\left(= 0.74^2\right) = 0.55$ ***A1***

which is greater than 0.5 ***AG***

[6 marks]

(b) (i) $\mathrm{P}(X \le 1) = 0.3296\ldots$ ***(A1)***

$\mathrm{P}(1 \le X \le 5) = 0.5349\ldots$ ***(A1)***

$\mathrm{P}(5 \le X \le 10) = 0.1170\ldots$ ***(A1)***

$\mathrm{E}(\text{score}) = 10 \times 0.3296\ldots + 5 \times 0.5349\ldots + 1 \times 0.1170\ldots$ ***M1A1***

$= 6.09$ ***A1***

Note:	Accept probabilities in exponential form until the final decimal answer.

(ii) E (score) for X with unknown parameter can be expressed as $10 \times (1 - e^{-\lambda}) + 5 \times (e^{-\lambda} - e^{-5\lambda}) + (e^{-5\lambda} - e^{-10\lambda})$ ***(M1)(A1)***

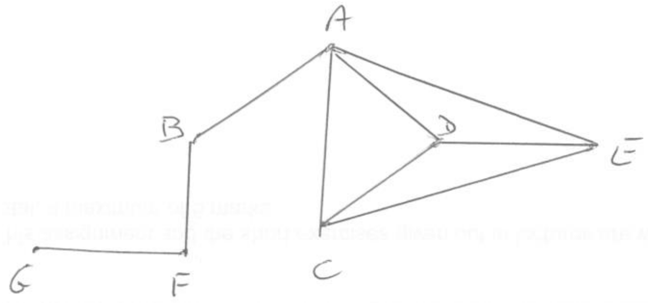
attempt to solve $\mathrm{E}(\text{score}) = 6.5$ ***(M1)***

obtain $\lambda = 0.473$ ***A1***

[10 marks]

Total [16 marks]

4. (a) (i)



A2

Note: Award ***A1*** if one error made.

(ii) 4 *A1*

[3 marks]

(b) (i) yes, for example GFBACDE *A1R1*

(ii) no, for example F and B would be visited twice *A1R1*

(iii) no, because the graph contains vertices of odd degree *A1R1*

(iv) no, because there are more than two vertices of odd degree *A1R1*

Note: The ***A*** and ***R*** marks can be considered as independent.

[8 marks]

(c) $v = 7$, $e = 9$ *A1*

$f = 4$ from (a)(ii)

$9 + 2 = 7 + 4$ *R1AG*

[2 marks]

(d) no, because the graph contains at least one triangle *A1R1*

[2 marks]

(e)

$$\begin{pmatrix} & A & B & C & D & E & F & G \\ A & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ B & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ C & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ D & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ E & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ F & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ G & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

A2

Note: ***A1*** for one error, ***A0*** for more than one error.

[2 marks]

(f) **METHOD 1**

DG element of 7^{th} power of matrix $=26$ ***M1A1A1***

Note: *M1* for attempt at some power; *A1* for 7^{th} power; *A1* for 26.

DG element of the 5^{th} power of the matrix $=2$ ***A1A1***
obtain $26-2=24$ ***M1A1***

METHOD 2

the observation that letter has to reach Grace after Frank obtains it after 6 passings, (without Grace having received it earlier) ***(M1)(A1)***
statement that the G row and column have been deleted ***A1A1***
DF element of 6th power of new matrix is 24 ***M1A1A1***

Note: *M1* for attempt at some power of new or old matrix; *A1* for 6^{th} power of new matrix; *A1* for 24.

[7 marks]

Total [24 marks]

5. (a) reflexive: if $r = a + b\sqrt{2} \in S$ then $a \equiv a$ (mod2) and $b \equiv b$ (mod3)
$(\Rightarrow rRr)$ ***A1***
symmetric: if $r_1 R r_2$, then $a_1 \equiv a_2$ (mod2) and $b_1 \equiv b_2$ (mod3), and ***M1***
$a_2 \equiv a_1$ (mod2) and $b_2 \equiv b_1$ (mod3), (so that $r_2 R r_1$) ***A1***
transitive: if $r_1 R r_2$ and $r_2 R r_3$ then
$2 \mid a_1 - a_2$ and $2 \mid a_2 - a_3$ ***M1***
$\Rightarrow 2 \mid a_1 - a_2 + a_2 - a_3 \Rightarrow 2 \mid a_1 - a_3$ ***M1A1***
$3 \mid b_1 - b_2$ and $3 \mid b_2 - b_3$
$\Rightarrow 3 \mid b_1 - b_2 + b_2 - b_3 \Rightarrow 3 \mid b_1 - b_3$ $(\Rightarrow r_1 R r_3)$ ***A1AG***

[7 marks]

(b) consider, for example, $r_1 = 1 + \sqrt{2}$, $r_2 = 3 + \sqrt{2}$, $(r_1 R r_2)$ ***M1***

Note: Only award ***M1*** if the two numbers are related and neither a nor $b = 0$.

$r_1^2 = 3 + 2\sqrt{2}$, $r_2^2 = 11 + 6\sqrt{2}$ ***A1***
the squares are not equivalent because $2 \neq 6$ (mod3) ***A1***

[3 marks]

(c) (i) $E = \left\{ 2k + 1 + (3m+1)\sqrt{2} : k, m \in \mathbb{Z} \right\}$ ***A1A1***

(ii) $F = \left\{ 2k + 1 + (3m-1)\sqrt{2} : k, m \in \mathbb{Z} \right\}$ ***A1***

[3 marks]

(d) (i) $\left(1 + \sqrt{2}\right)^3 = 7 + 5\sqrt{2}$ ***A1***
$= 2 \times 3 + 1 + (3 \times 2 - 1)\sqrt{2} \in F$ ***R1AG***

(ii) $\left(1 + \sqrt{2}\right)^6 = 99 + 70\sqrt{2}$ ***A1***
$= 2 \times 49 + 1 + (3 \times 23 + 1)\sqrt{2} \in E$ ***R1AG***

[4 marks]

(e) (i) E is not a group under addition ***A1***
any valid reason *eg* $0 \notin E$ ***R1***

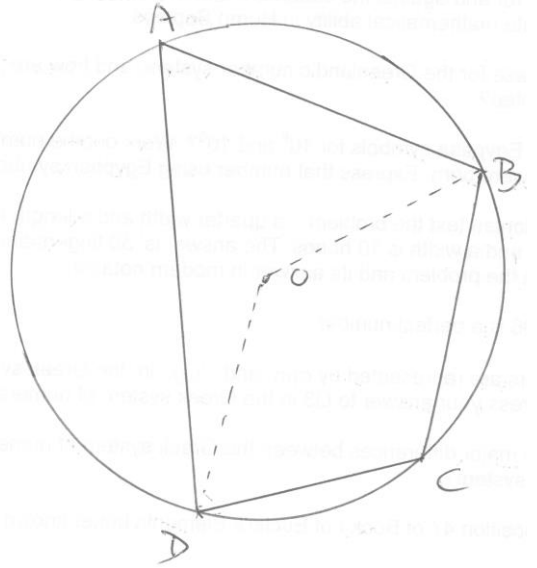
(ii) E is not a group under multiplication ***A1***
any valid reason *eg* $1 \notin E$ ***R1***

[4 marks]

Total [21 marks]

6. Part A

(a)



recognition of relevant theorem *(M1)*

eg $\hat{DOB} = 2 \times \hat{DAB}$ ***A1***

$360^\circ - \hat{DOB} = 2 \times \hat{DCB}$ ***A1***

so $\hat{DAB} + \hat{DCB} = 180^\circ$ ***AG***

[3 marks]

continued ...

Question 6 continued

(b)

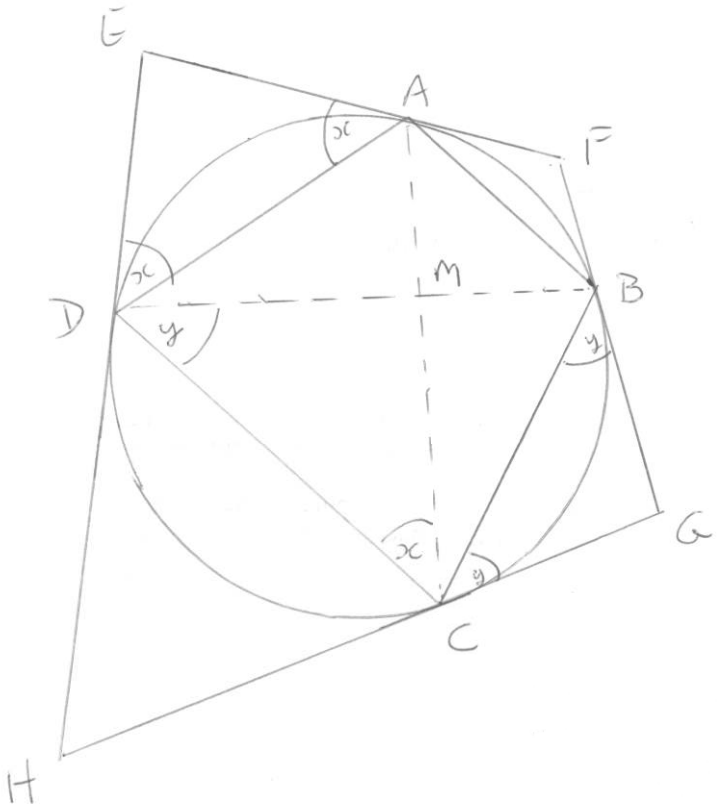


diagram showing tangents EAF, FBG, GCH and HDE; diagonals cross at M. ***M1***

let $x = \text{E}\hat{\text{D}}\text{A} = \text{E}\hat{\text{A}}\text{D}$; $y = \text{B}\hat{\text{C}}\text{G} = \text{C}\hat{\text{B}}\text{G}$ ***A1***

$\text{D}\hat{\text{E}}\text{A} + \text{H}\hat{\text{G}}\text{F} = 180 - 2x + 180 - 2y = 360 - 2(x+y)$ ***M1A1***

$\text{C}\hat{\text{D}}\text{B} = y$ and $\text{A}\hat{\text{C}}\text{D} = x$, as angles in alternate segments ***M1A1***

$\text{D}\hat{\text{M}}\text{C} = 180 - (x+y) = \left(\frac{1}{2}\right)(\text{D}\hat{\text{E}}\text{A} + \text{H}\hat{\text{G}}\text{F}) = 90^\circ$ ***A1***

so the diagonals cross at right angles ***AG***

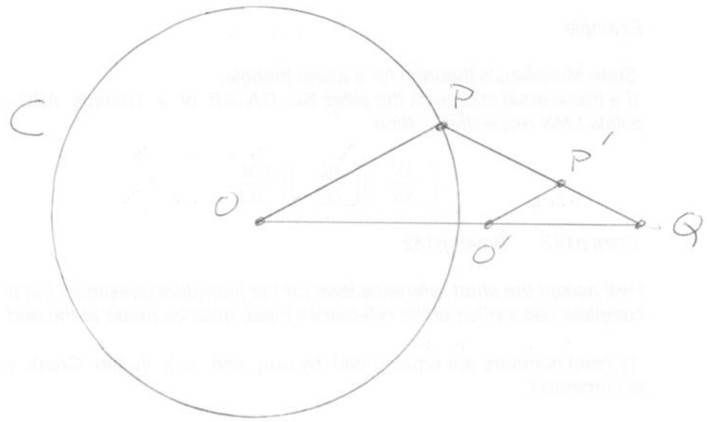
[7 marks]

continued ...

Question 6 continued

Part B

(a)



	M1
let $\mathrm{O'}$ be the point on OQ such $\mathrm{O'P'}$ is parallel to OP	***A1***
using similar triangles $\mathrm{O'Q} = k\mathrm{OQ}$, so $\mathrm{O'}$ is a fixed point	***M1A1***
and $\mathrm{O'P'} = k\mathrm{OP}$ which is constant	***A1***
so $\mathrm{P'}$ lies on a circle centre $\mathrm{O'}$	***R1***
so the locus of $\mathrm{P'}$ is a circle	***AG***

[6 marks]

(b)

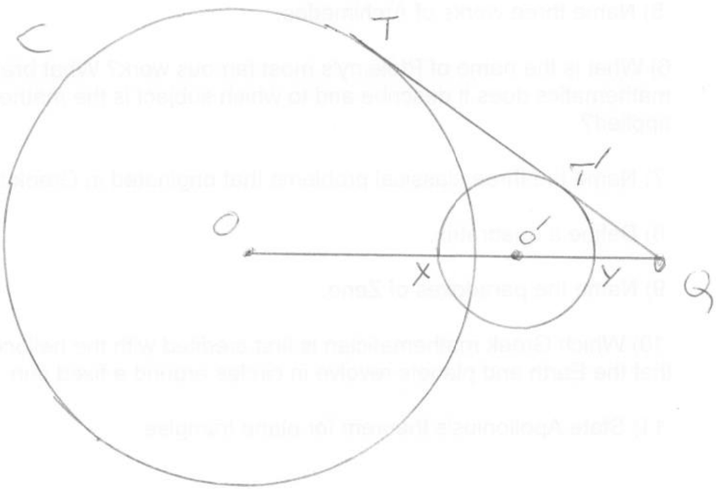
let one of the two tangents to C from Q touch C at T	
the image of T lies on TQ	***A1***
and is a unique point $\mathrm{T'}$ on C'	***A1***
so $\mathrm{TT'}$ is a common tangent and passes through Q	***R1***
the same is true for the other tangent	***A1***
so the two tangents to C from Q are also tangents to C'	***AG***

[4 marks]

continued …

Question 6 continued

(c)



by the tangent-secant theorem for C', $\text{QX} \times \text{QY} = \text{QT}'^2$ ***M2***

$= k^2\text{QT}^2$ ***A1***

$k^2 p$ using one of the various definitions of power ***AG***

[3 marks]

Total [23 marks]